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A Simple Proof of the Equivalence of the Limiting Distributions of the Continuous-Time and the Embedded Process of the Queue Size in the M/G/1 Queue

by

A. Hordijk & H.C. Tijms

ABSTRACT

This paper presents an extremely simple proof of the known remarkable fact that for the M/G/1 queue the continuous-time process describing the number of customers in the system has the same limiting distribution as the embedded process describing the number of customers in the system just after service completion epochs.

KEY WORDS & PHRASES: M/G/1 *queue, equivalence of limiting distributions.*

1. INTRODUCTION

We consider the M/G/1 queue where customers arrive in accordance with a Poisson process with rate λ and the service times of the customers are independent random variables having a common probability distribution function F with finite mean μ. It is assumed that $\rho = \lambda\mu < 1$. The purpose of this paper is to present an extremely simple proof of the known remarkable result that the continuous-time process describing the number of customers in the system and the embedded process describing the number of customers present just after service completion epochs have the same limiting distributions. In COHEN [1] this result was proved by calculating each of these distributions and verifying that they are the same, cf. also COHEN [2] for a more simple approach. Another proof of the above result may be obtained by using the rather deep Theorem 3 in STIDHAM [4] and by observing that the embedded process describing the number of customers present just before arrival epochs has the same limiting distribution as the one describing the number of customers present just after service completion epochs. The proof that will be given in this paper is short and simple and is based on a well-known standard result in the theory of regenerative processes. For other applications of this powerful theory to queueing problems we refer to COHEN [2] and STIDHAM [4]. In particular COHEN [2] gives a large number of elegant derivations of known results in queueing theory by using the theory of regenerative processes.

2. PROOF

For ease we assume throughout this paper that at epoch 0 a service is completed and the system becomes empty. Denote by T the next epoch at which a service is completed and the system becomes empty. Let N be the number of customers served in the busy cycle (0,T]. By a simple standard argument from busy period analysis the following well-known results are easily obtained (cf. COHEN [1]),

(1) $ET = 1/\lambda(1-\rho)$ and $EN = 1/(1-\rho)$.

For $j = 0,1,\ldots$, let T_j be the amount of time during which j customers are in the system in (0,T] and let N_j be the number of service completion epochs at which j customers are left behind in (0,T]. Denote by $X(t)$, $t \geq 0$ the number of customers in the system at epoch t and, for $n = 0,1,\ldots$, let X_n be the number of customers present just after the nth service completion epoch (the 0th service completion occurs at epoch 0).

By a basic result in the theory of regenerative processes (see SMITH [3] and STIDHAM [4]), we have that, for all $j = 0,1,\ldots$, the limits

$$v_j = \lim_{t\to\infty} \Pr\{X(t) = j\} \quad \text{and} \quad \pi_j = \lim_{n\to\infty} \Pr\{X_n = j\}$$

exist and are given by

$$(2) \qquad v_j = ET_j/ET \quad \text{and} \quad \pi_j = EN_j/EN \qquad \text{for all } j = 0,1,\ldots .$$

Further, both $\{v_j\}$ and $\{\pi_j\}$ are probability distributions. A famous result in queueing theory states (cf. p.247 in COHEN [1])

$$(3) \qquad v_j = \pi_j \qquad \text{for all } j = 0,1,\ldots .$$

We shall now give a simple derivation of (3) by using (2). By (1)-(2), we have that (3) is equivalent to

$$(4) \qquad \lambda ET_j = EN_j \qquad \text{for all } j = 0,1,\ldots .$$

We shall give now two proofs of (4). The first one proceeds as follows. Since $ET_0 = 1/\lambda$ and $EN_0 = 1$ we have that (4) holds for $j = 0$. Also, observe that, by (1)-(2),

$$(5) \qquad \pi_0 = 1 - \rho.$$

To verify (4), we first introduce some notation. For $k = 0,1,\ldots$, let

$$p_k = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} dF(t) \quad \text{and} \quad q_k = \sum_{j=k+1}^{\infty} p_j,$$

i.e. p_k is the probability of k arrivals during the service time of a customer. Further, let τ and S be independent random variables which are distributed as the interarrival time between two successive arrivals and the service time of a customer, respectively. We easily get

$$(6) \qquad \mathrm{E}\min(\tau,S) = \frac{1}{\lambda}\int_0^\infty (1-e^{-\lambda t})dF(t) = \frac{1}{\lambda}(1-p_0).$$

Now, by using Wald's equation and the well-known fact that given that n arrivals occurred in (0,t) the n arrival epochs have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on (0,t) (e.g. p. 70 in COHEN [1]), it is easily seen that

$$(7) \qquad ET_n = \int_0^\infty \left\{\sum_{k=n-1}^\infty \frac{t}{k+1} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\right\} dF(t) + EN_n \mathrm{E}\min(\tau,S) +$$

$$+ \sum_{j=1}^{n-1} EN_j \int_0^\infty \left\{\sum_{k=n-j}^\infty \frac{t}{k+1} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\right\} dF(t) \qquad \text{for all } n \geq 1.$$

The first term in the right side of (7) arises from the service time of the first customer served in (0,T]. Under the condition that this service time is t and that $k \geq n-1$ arrivals occur during this time t, we get from this service time a contribution of $t/(k+1)$ to ET_n. Similarly, we get the other terms in the right side of (7). By (6) and the fact that $q_0 = 1 - p_0$, we can write (7) as

$$(8) \qquad \lambda ET_n = \sum_{j=1}^{n} EN_j\, q_{n-j} + q_{n-1} \qquad \text{for all } n \geq 1.$$

We now introduce the following generating functions for $|s| \leq 1$,

$$T(s) = \sum_{n=1}^\infty ET_n s^n,\ N(s) = \sum_{n=1}^\infty EN_n s^n,\ P(s) = \sum_{n=0}^\infty p_n s^n,$$

$$Q(s) = \sum_{n=0}^\infty q_n s^n\ ,\ \Pi(s) = \sum_{n=0}^\infty \pi_n s^n.$$

As already noted $\lambda ET_0 = EN_0$, so (4) holds when we can show that $\lambda T(s) = N(s)$ for $|s| < 1$. From (8), it follows that $\lambda T(s) = N(s)Q(s) + sQ(s)$ for $|s| \leq 1$. Hence relation (4) follows by verifying that $N(s) = N(s)Q(s) + sQ(s)$ for all $|s| < 1$, or

$$(9) \qquad N(s) = sQ(s)/\{1 - Q(s)\} \qquad \text{for } |s| < 1.$$

Now, by using (1), (3) and the steady state equation $\pi_n = \pi_0 p_n + \sum_{k=1}^{n+1} \pi_k p_{n-k+1}$, $n \geq 0$ for the Markov chain $\{X_n\}$, we have for $|s| < 1$

$$1+N(s) = \Pi(s)/(1-\rho),\ \Pi(s) = \pi_0\{sP(s)-P(s)\}/\{s-P(s)\},\ Q(s) = \{1-P(s)\}/(1-s).$$

Together these relations and (5) verify (9) which completes the proof.

We now give another proof of (4). This proof which was suggested by Prof. J.W. Cohen proceeds as follows. Clearly, EN_j can be interpreted as the expected number of downcrossings of the process $\{X(t)\}$ to state j during $(0,T]$. Since this expectation equals the expected number of upcrossings of the process $\{X(t)\}$ to state $j+1$ during $(0,T]$, relation (4) follows when we can show that, for all $j = 0,1,\ldots,$

(10) λET_j = expected number of upcrossings of the process $\{X(t)\}$ to state $j+1$ during $(0,T]$.

To prove this, define for any $u > 0$ and $j = 0,1,\ldots,$ $K_j(u)$ = number of upcrossings of the process $\{X(t)\}$ to state $j+1$ during $(0,u]$ and $A_j(u)$ = amount of time during which j customers are present in $(0,u]$. Then, by virtue of the fact that the arrival process is a Poisson process, we have for all $j = 0,1,\ldots,$

(11) $EK_j(t) = \lambda EA_j(t)$ for all $t > 0$.

Now, by a basic result in the theory of regenerative processes (see [3] and [4]), we have for all $j = 0,1,\ldots,$

(12) $\lim_{t\to\infty} EK_j(t)/t = EK_j(T)/ET$ and $\lim_{t\to\infty} EA_j(t)/t = EA_j(T)/ET = ET_j/ET$.

Together (11) and (12) prove (10) which ends the proof.

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